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Regularization proximal point algorithm for finding a common fixed point of a finite family of nonexpansive mappings in Banach spaces

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available at the end of the article**Abstract**

We study the strong convergence of a regularization proximal point algorithm for finding a common fixed point of a finite family of nonexpansive mappings in a uniformly convex and uniformly smooth Banach space.

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1 Introduction

Let E be a Banach space with its dual space E^* . For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|\cdot\|$. We write $\langle x, x^* \rangle$ instead of $x^*(x)$ for $x^* \in E^*$ and $x \in E$. We denote as \rightharpoonup and \rightarrow , the weak convergence and strong convergence, respectively. A Banach space E is reflexive if $E = E^{**}$.

The problem of finding a fixed point of a nonexpansive mapping is equivalent to the problem of finding a zero of the following operator equation:

$$0 \in A(x) \quad (1.1)$$

involving the accretive mapping A .

One popular method of solving equation $0 \in A(x)$ is the proximal point algorithm of Rockafellar [1] which is recognized as a powerful and successful algorithm for finding a zero of monotone operators. Starting from any initial guess $x_0 \in H$, this proximal point algorithm generates a sequence $\{x_n\}$ given by

$$x_{n+1} = J_{c_n}^A(x_n + e_n), \quad (1.2)$$

where $J_r^A = (I + rA)^{-1}$, $\forall r > 0$ is the resolvent of A in a Hilbert space H . Rockafellar [1] proved the weak convergence of the algorithm (1.2) provided that the regularization sequence $\{c_n\}$ remains bounded away from zero, and that the error sequence $\{e_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \|e_n\| < \infty$. However, Güler's example [2] shows that proximal point algorithm (1.2) has only weak convergence in an infinite-dimensional Hilbert space. Recently, several authors proposed modifications of Rockafellar's proximal point algorithm (1.2) for the strong convergence. For example, Solodov and Svaiter [3] and

Kamimura and Takahashi [4] studied a modified proximal point algorithm by an additional projection at each step of iteration. Lehdili and Moudafi [5] obtained the convergence of the sequence $\{x_n\}$ generated by the algorithm:

$$x_{n+1} = J_{C_n}^{A_n}(x_n), \quad (1.3)$$

where $A_n = \mu_n I + A$ is viewed as a Tikhonov regularization of A . When A is maximal monotone in a Hilbert space H , Xu [6], Song and Yang [7] used the technique of non-expansive mappings to get convergence theorems for $\{x_n\}$ defined by the perturbed version of the algorithm (1.3):

$$x_{n+1} = J_{r_n}^A(t_n u + (1 - t_n)x_n). \quad (1.4)$$

The equation (1.4) can be written in the following equivalent form:

$$r_n A(x_{n+1}) + x_{n+1} \ni t_n u + (1 - t_n)x_n. \quad (1.5)$$

In this article, we study a regularization proximal point algorithm to solve the problem of finding a common fixed point of a finite family of nonexpansive self-mappings in a uniformly convex and uniformly smooth Banach space E . Moreover, we give some analogue regularization methods for the more general problems, such as: problem of finding a common fixed point of a finite family of nonexpansive mappings T_i , $i = 1, 2, \dots, N$, where T_i is self-mapping or nonself-mapping on a closed convex subset of E .

2 Preliminaries

Definition 2.1. A Banach space E is said to be uniformly convex, if for any $\varepsilon \in (0, 2]$ the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$, $\|x - y\| \geq \varepsilon$ imply that there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\frac{\|x + y\|}{2} \leq 1 - \delta.$$

The function

$$\delta_E(\varepsilon) = \inf\{1 - 2^{-1} \|x + y\| : \|x\| = \|y\| = 1, \|x - y\| = \varepsilon\} \quad (2.1)$$

is called the modulus of convexity of the space E . The function $\delta_E(\varepsilon)$ defined on the interval $[0, 2]$ is continuous, increasing and $\delta_E(0) = 0$. The space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$, $\forall \varepsilon \in (0, 2]$.

The function

$$\rho_E(\tau) = \sup\{2^{-1}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1, \|y\| = \tau\}, \quad (2.2)$$

is called the modulus of smoothness of the space E . The function $\rho_E(\tau)$ defined on the interval $[0, +\infty)$ is convex, continuous, increasing and $\rho_E(0) = 0$.

Definition 2.2. A Banach space E is said to be uniformly smooth, if

$$\lim_{\tau \rightarrow 0} \frac{\rho_E(\tau)}{\tau} = 0. \quad (2.3)$$

It is well known that every uniformly convex and uniformly smooth Banach space is reflexive. In what follows, we denote

$$h_E(\tau) = \frac{\rho_E(\tau)}{\tau}. \quad (2.4)$$

The function $h_E(\tau)$ is nondecreasing. In addition, it is not difficult to show that the estimate

$$h_E(K\tau) \leq LKh_E(\tau), \quad \forall K > 1, \quad \tau > 0, \quad (2.5)$$

is valid, where L is the Figiel's constant [8-10], $1 < L < 1.7$. Indeed, we know that the inequality holds ([8])

$$\frac{\rho_E(\eta)}{\eta^2} \leq L \frac{\rho_E(\xi)}{\xi^2}, \quad \forall \eta \geq \xi > 0. \quad (2.6)$$

It implies that

$$\xi h_E(\eta) \leq L\eta h_E(\xi), \quad \forall \eta \geq \xi > 0. \quad (2.7)$$

Taking in (2.7) $\eta = C\tau$ and $\xi = \tau$, we obtain the inequality:

$$\tau h_E(C\tau) \leq LC\tau h_E(\tau), \quad (2.8)$$

which implies that (2.5) holds. Similarly, we have

$$\rho_E(C\tau) \leq LC^2 \rho_E(\tau), \quad \forall C > 1, \quad \tau > 0. \quad (2.9)$$

Definition 2.3. A mapping j from E onto E^* satisfying the condition

$$j(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 \text{ and } \|f\| = \|x\|\} \quad (2.10)$$

is called the normalized duality mapping of E .

We know that

$$j(x) = 2^{-1} \text{grad } \|x\|^2.$$

in a smooth Banach space, and the normalized duality mapping J is the identity operator I in a Hilbert space.

Definition 2.4. An operator $A : D(A) \subseteq E \rightarrow E$ is called accretive, if for all $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle A(x) - A(y), j(x - y) \rangle \geq 0. \quad (2.11)$$

Definition 2.5. An operator $A : E \rightarrow E$ is called m -accretive if it is an accretive operator and the range $R(\lambda A + I) = E$ for all $\lambda > 0$, where I is the identity of E .

If A is an m -accretive operator then it is a demiclosed operator, i.e., if the sequence $\{x_n\} \subset D(A)$ satisfies $x_n \rightarrow x$ and $A(x_n) \rightarrow f$, then $A(x) = f$ [10,11].

Definition 2.6. A mapping $T : C \rightarrow E$ is said to be nonexpansive on a closed convex subset C of Banach space E if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (2.12)$$

If $T : C \rightarrow E$ is a nonexpansive then $I - T$ is an accretive operator. In this case, if the subset C coincides E then $I - T$ is an m -accretive operator.

Definition 2.7. Let G be a nonempty closed convex subset of E . A mapping $Q_G : E \rightarrow G$ is said to be

- (i) a retraction onto G if $Q_G^2 = Q_G$;

(ii) a nonexpansive retraction if it also satisfies the inequality:

$$\|Q_G x - Q_G y\| \leq \|x - y\|, \quad \forall x, y \in E; \quad (2.13)$$

(iii) a sunny retraction if for all $x \in E$ and for all $t \in [0, +\infty)$

$$Q_G(Q_G x + t(x - Q_G x)) = Q_G x. \quad (2.14)$$

A closed convex subset C of E is said to be a nonexpansive retract of E , if there exists a nonexpansive retraction from E onto C , and it is said to be a sunny nonexpansive retract of E , if there exists a sunny nonexpansive retraction from E onto C .

Proposition 2.8. [9] *Let G be a nonempty closed convex subset of E . A mapping $Q_G : E \rightarrow G$ is a sunny nonexpansive retraction if and only if*

$$\langle x - Q_G x, J(\xi - Q_G x) \rangle \leq 0, \quad \forall x \in E, \forall \xi \in G. \quad (2.15)$$

Reich [12] showed that if E is uniformly smooth and D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D , and it can be constructed as follows.

Lemma 2.9. [12] *Let E be a uniformly smooth Banach space, and let $T : C \rightarrow C$ be a nonexpansive mapping with a fixed point. For each $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converges strongly as $t \rightarrow 0$ to a fixed point of T . Define $Q : C \rightarrow \text{Fix}(T)$ by $Qu = \lim_{t \rightarrow 0} x_t$. Then, Q is a unique sunny nonexpansive retraction from C onto $\text{Fix}(T)$, i.e., Q satisfies the property:*

$$\langle u - Qu, J(z - Qu) \rangle \leq 0, \quad u \in C, z \in \text{Fix}(T). \quad (2.16)$$

Definition 2.10. Let C_1 and C_2 be convex subsets of E . The quantity

$$\beta(C_1, C_2) = \sup_{u \in C_1} \inf_{v \in C_2} \|u - v\| (= \sup_{u \in C_1} d(u, C_2))$$

is said to be a semideviation of the set C_1 from the set C_2 . The function

$$\mathcal{H}(C_1, C_2) = \max\{\beta(C_1, C_2), \beta(C_2, C_1)\}$$

is said to be a Hausdorff distance between C_1 and C_2 .

In this article, we will use the following useful lemma:

Lemma 2.11. [7] *If E is a uniformly smooth Banach space, C_1 and C_2 are closed convex subsets of E such that the Hausdorff distance $\mathcal{H}(C_1, C_2) \leq \delta$, and Q_{C_1} and Q_{C_2} are the sunny nonexpansive retractions onto the subsets C_1 and C_2 , respectively, then*

$$\|Q_{C_1} x - Q_{C_2} x\|^2 \leq 16R(2r + d)h_E \left(\frac{16L\delta}{R} \right), \quad (2.17)$$

where L is Figiel's constant, $r = \|x\|$, $d = \max\{d_1, d_2\}$, and $R = 2(2r + d) + \delta$. Here $d_i = \text{dist}(\theta, C_i) = d(\theta, C_i)$, $i = 1, 2$, and θ is the origin of the space E .

3 Main results

We need the following lemmas in the proof of our results:

Lemma 3.1. [9] *If $A = I - T$ with a nonexpansive mapping T , then for all $x, y \in D(T)$, the domain of T*

$$\langle Ax - Ay, J(x - y) \rangle \geq L^{-1}R^2\delta_E \left(\frac{\|Ax - Ay\|}{4R} \right), \quad (3.1)$$

where $\|x\| \leq R$, $\|y\| \leq R$ and $1 < L < 1.7$ is Figiel's constant.

Lemma 3.2. [13] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\beta_n + \sigma_n, \quad \forall n \geq 0$$

where $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\sigma_n\}$ satisfy the following conditions.

- (i) $\sum_{n=0}^{\infty} \lambda_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \beta_n \leq 0$ or $\sum_{n=0}^{\infty} |\lambda_n\beta_n| < \infty$;
- (iii) $\sigma_n \geq 0 \quad \forall n \geq 0$ and $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then, $\{a_n\}$ converges to zero.

Lemma 3.3. [9] Let E be a uniformly smooth Banach space. Then, for all $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + c\rho_E(\|y\|), \quad (3.2)$$

where $c = 48 \max(L, \|x\|, \|y\|)$.

First, we consider the following problem:

$$\text{Finding an element } x^* \in S = \bigcap_{i=1}^N \text{Fix}(T_i), \quad (3.3)$$

where $\text{Fix}(T_i)$ is the set of fixed points of the nonexpansive mapping $T_i : E \rightarrow E$, $i = 1, 2, \dots, N$.

Theorem 3.4. Suppose that E is a uniformly convex and uniformly smooth Banach space which has a weakly sequentially continuous normalized duality mapping j from E to E^* . Let $T_i : E \rightarrow E$, $i = 1, 2, \dots, N$ be nonexpansive mappings with $S = \bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy

- (i) $\lim_{n \rightarrow \infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
- (ii) $\lim_{n \rightarrow \infty} r_n = +\infty$,

then the sequence $\{x_n\}$ defined by

$$r_n \sum_{i=1}^N A_i(x_{n+1}) + x_{n+1} = t_n u + (1 - t_n)x_n, \quad u, x_0 \in E, \quad n \geq 0 \quad (3.4)$$

converges strongly to $Q_S u$, where $A_i = I - T_i$, $i = 1, 2, \dots, N$ and Q_S is a sunny nonexpansive retraction from E onto S .

Proof. First, equation (3.4) defines a unique sequence $\{x_n\} \subset E$, because for each n , the element x_{n+1} is a unique fixed point of the contraction mapping $f : E \rightarrow E$ defined by

$$f(x) = \frac{r_n}{Nr_n + 1} \sum_{i=1}^N T_i(x) + \frac{1}{Nr_n + 1} [t_n u + (1 - t_n)x_n], \quad x \in E.$$

For every $x^* \in S$, we have

$$\langle r_n \sum_{i=1}^N A_i(x_{n+1}), j(x_{n+1} - x^*) \rangle \geq 0, \quad \forall n \geq 0. \quad (3.5)$$

Therefore,

$$\langle t_n u + (1 - t_n)x_n - x_{n+1}, j(x_{n+1} - x^*) \rangle \geq 0, \quad \forall n \geq 0. \quad (3.6)$$

It gives the inequality as follows:

$$\|x_{n+1} - x^*\|^2 \leq \{t_n \|u - x^*\| + (1 - t_n) \|x_n - x^*\|\} \times \|x_{n+1} - x^*\|.$$

Consequently, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq t_n \|u - x^*\| + (1 - t_n) \|x_n - x^*\| \\ &\leq \max(\|u - x^*\|, \|x_n - x^*\|) \\ &\vdots \\ &\leq \max(\|u - x^*\|, \|x_0 - x^*\|), \quad \forall n \geq 0. \end{aligned}$$

Therefore, the sequence $\{x_n\}$ is bounded. Every bounded set in a reflexive Banach space is relatively weakly compact. This means that there exists a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ which converges to a limit $\bar{x} \in E$.

Suppose $\|x_n\| \leq R$ and $\|x^*\| \leq R$ with $R > 0$. By Lemma 3.1, we have

$$\begin{aligned} \delta_E \left(\frac{\|A_i(x_{n+1})\|}{4R} \right) &\leq \frac{L}{R^2 r_n} \langle r_n A_i(x_{n+1}), j(x_{n+1} - x^*) \rangle \\ &\leq \frac{L}{R^2 r_n} \langle r_n \sum_{k=1}^N A_k(x_{n+1}), j(x_{n+1} - x^*) \rangle \\ &\leq \frac{L}{R^2 r_n} \|t_n u + (1 - t_n)x_n - x_{n+1}\| \cdot \|x_{n+1} - x^*\| \\ &\rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

for every $i = 1, 2, \dots, N$. Since the modulus of convexity δ_E is continuous and E is a uniformly convex Banach space, $A_i(x_{n+1}) \rightarrow 0$, $i = 1, 2, \dots, N$. It is clear that $\bar{x} \in S$ from the demiclosedness of A_i . Hence, noting the inequality (2.15), we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - Q_S u, j(x_n - Q_S u) \rangle &= \lim_{k \rightarrow \infty} \langle u - Q_S u, j(x_{n_k} - Q_S u) \rangle \\ &= \langle u - Q_S u, j(\bar{x} - Q_S u) \rangle \\ &\leq 0. \end{aligned} \quad (3.7)$$

Next, we have

$$\begin{aligned} \|x_{n+1} - Q_S u\|^2 &= \langle -r_n \sum_{i=1}^N A_i(x_{n+1}) + t_n u + (1 - t_n)x_n - Q_S u, J(x_{n+1} - Q_S u) \rangle \\ &= -\langle r_n \sum_{i=1}^N A_i(x_{n+1}), J(x_{n+1} - Q_S u) \rangle \\ &\quad + \langle t_n u + (1 - t_n)x_n - Q_S u, J(x_{n+1} - Q_S u) \rangle \\ &\leq \langle t_n(u - Q_S u) + (1 - t_n)(x_n - Q_S u), J(x_{n+1} - Q_S u) \rangle \\ &\leq \frac{1}{2} \{ \|t_n(u - Q_S u) + (1 - t_n)(x_n - Q_S u)\|^2 + \|x_{n+1} - Q_S u\|^2 \}. \end{aligned}$$

By the Lemma 3.3 and the above inequality, we conclude that

$$\begin{aligned} \|x_{n+1} - Q_S u\|^2 &\leq \|t_n(u - Q_S u) + (1 - t_n)(x_n - Q_S u)\|^2 \\ &\leq (1 - t_n)^2 \|x_n - Q_S u\|^2 + 2t_n(1 - t_n) \langle u - Q_S u, j(x_n - Q_S u) \rangle \\ &\quad + c\rho_E(t_n \|u - Q_S u\|). \end{aligned}$$

Consequently, we have

$$\|x_{n+1} - Q_S u\|^2 \leq (1 - t_n) \|x_n - Q_S u\|^2 + t_n \beta_n, \quad (3.8)$$

where

$$\beta_n = 2(1 - t_n) \langle u - Q_S u, j(x_n - Q_S u) \rangle + c \frac{\rho_E(t_n \|u - Q_S u\|)}{t_n}.$$

Since E is a uniformly smooth Banach space, $\frac{\rho_E(t_n \|u - Q_S u\|)}{t_n} \rightarrow 0$, $n \rightarrow \infty$. By (3.7), we obtain $\limsup_{n \rightarrow \infty} \beta_n \leq 0$. Hence, an application of Lemma 3.2 on (3.8) yields the desired result. \square

Now, we will give a method to solve more generally following problem:

$$\text{Finding an element } x^* \in S = \bigcap_{i=1}^N \text{Fix}(T_i), \quad (3.9)$$

where $T_i : C_i \rightarrow C_i$, $i = 1, 2, \dots, N$ is a nonexpansive mapping and C_i is a convex closed nonexpansive retract of E .

Obviously, we have the following lemma:

Lemma 3.5. *Let E be a Banach space, and let C be a closed convex retract of E . Let $T : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Then, $\text{Fix}(T) = \text{Fix}(TQ_C)$, where Q_C is a retraction of E onto C .*

We consider the iterative sequence $\{x_n\}$ defined by

$$r_n \sum_{i=1}^N B_i(x_{n+1}) + x_{n+1} = t_n u + (1 - t_n)x_n, \quad u, x_0 \in E, \quad n \geq 0, \quad (3.10)$$

where $B_i = I - T_i Q_{C_i}$, $i = 1, 2, \dots, N$ and Q_{C_i} is a nonexpansive retraction from E onto C_i , $i = 1, 2, \dots, N$.

Theorem 3.6. *Suppose that E is a uniformly convex and uniformly smooth Banach space which has a weakly sequentially continuous normalized duality mapping j from E into E^* . Let C_i be a convex closed nonexpansive retract of E and let $T_i : C_i \rightarrow C_i$, $i = 1, 2, \dots, N$ be a nonexpansive mapping such that $S = \bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

- (i) $\lim_{n \rightarrow \infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
- (ii) $\lim_{n \rightarrow \infty} r_n = +\infty$,

then the sequence $\{x_n\}$ generated by (3.10) converges strongly to $Q_S u$, where Q_S is a sunny nonexpansive retraction from E onto S .

Proof. By the Lemma 3.5, we have $S = \bigcap_{i=1}^N \text{Fix}(T_i Q_{C_i})$ and applying Theorem 3.4, we obtain the proof of this Theorem. \square

Next, we study the stability of the regularization algorithm (3.10) in the case that each C_i is a closed convex sunny nonexpansive retract of E with respect to perturbations of operators T_i and constraints C_i , $i = 1, 2, \dots, N$ satisfying following conditions:

(P1) Instead of C_i , there is a sequence of closed convex sunny nonexpansive retracts $C_i^n \subset E$, $n = 1, 2, 3, \dots$ such that

$$\mathcal{H}(C_i^n, C_i) \leq \delta_n, \quad i = 1, 2, \dots, N,$$

where $\{\delta_n\}$ is a sequence of positive numbers.

(P2) For each set C_i^n , there is a nonexpansive self-mapping $T_i^n : C_i^n \rightarrow C_i^n$, $i = 1, 2, \dots, N$ satisfying the conditions: if for all $t > 0$, there exists the increasing positive functions $g(t)$ and $\xi(t)$ such that $g(0) \geq 0$, $\xi(0) = 0$ and $x \in C_i, y \in C_i^n$, $\|x - y\| \leq \delta$, then

$$\|T_i x - T_i^n y\| \leq g(\max\{\|x\|, \|y\|\})\xi(\delta). \quad (3.11)$$

We establish the convergence and stability of the regularization method (3.10) in the form:

$$r_n \sum_{i=1}^N B_i^n(z_{n+1}) + z_{n+1} = t_n u + (1 - t_n)z_n, \quad u, z_0 \in E, \quad n \geq 0, \quad (3.12)$$

where $B_i^n = I - T_i^n Q_{C_i^n}$, $i = 1, 2, \dots, N$ and $Q_{C_i^n}$ is a sunny nonexpansive retraction from E onto C_i^n , $i = 1, 2, \dots, N$.

Theorem 3.7. Suppose that E is a uniformly convex and uniformly smooth Banach space which has a weakly sequentially continuous normalized duality mapping j from E into E^* . Let C_i be a convex closed sunny nonexpansive retract of E and let $T_i : C_i \rightarrow C_i$, $i = 1, 2, \dots, N$ be nonexpansive mappings such that $S = \bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. If the conditions (P1) and (P2) are fulfilled, and the sequences $\{r_n\}$, $\{\delta_n\}$ and $\{t_n\}$ satisfy

- (i) $\lim_{n \rightarrow \infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
- (ii) $\lim_{n \rightarrow \infty} r_n = +\infty$;
- (iii) $\sum_{n=0}^{\infty} r_n \xi(a\sqrt{h_E(\delta_n)}) < +\infty$ for each $a > 0$,

then the sequence $\{z_n\}$ generated by (3.12) converges strongly to $Q_S u$, where Q_S is a sunny nonexpansive retraction from E onto S .

Proof. For each n , $\sum_{i=1}^N B_i^n$ is an m -accretive operator on E , so the equation (3.12) defines a unique element $z_{n+1} \in E$. From the equations (3.10) and (3.12), we have

$$\begin{aligned} & r_n \left\langle \sum_{i=1}^N B_i^n(z_{n+1}) - B_i^n(x_{n+1}), j(z_{n+1} - x_{n+1}) \right\rangle \\ & + r_n \left\langle \sum_{i=1}^N B_i^n(x_{n+1}) - B_i(x_{n+1}), j(z_{n+1} - x_{n+1}) \right\rangle + \|z_{n+1} - x_{n+1}\|^2 \\ & = (1 - t_n) \langle z_n - x_n, j(z_{n+1} - x_{n+1}) \rangle. \end{aligned} \quad (3.13)$$

By the accretivity of $\sum_{i=1}^N B_i^n$ and the equation (3.13), we deduce

$$\|z_{n+1} - x_{n+1}\| \leq (1 - t_n) \|z_n - x_n\| + r_n \sum_{i=1}^N \|B_i^n(x_{n+1}) - B_i(x_{n+1})\|. \quad (3.14)$$

For each $i \in \{1, 2, \dots, N\}$,

$$\|B_i^n(x_{n+1}) - B_i(x_{n+1})\| = \|T_i^n Q_{C_i^n} x_{n+1} - T_i Q_{C_i} x_{n+1}\|. \quad (3.15)$$

Since $\{x_n\}$ is bounded and $\mathcal{H}(C_i, C_i^n) \leq \delta_n$, there exist constants $K_{1,i} > 0$ and $K_{2,i} > 1$ such that

$$\|Q_{C_i^n} x_{n+1} - Q_{C_i} x_{n+1}\| \leq K_{1,i} \sqrt{h_E(K_{2,i} \delta_n)} \leq K_{1,i} \sqrt{K_{2,i} L} \sqrt{h_E(\delta_n)}. \quad (3.16)$$

By the condition (P2),

$$\|T_i^n Q_{C_i} x_{n+1} - T_i Q_{C_i} x_{n+1}\| \leq g(M_i) \xi(K_{1,i} \sqrt{K_{2,i} L} \sqrt{h_E(\delta_n)}), \quad (3.17)$$

where $M_i = \max\{\sup \|Q_{C_i} x_{n+1}\|, \sup \|Q_{C_i} x_{n+1}\|\} < +\infty$.

From (3.14), (3.15) and (3.17), we obtain

$$\|z_{n+1} - x_{n+1}\| \leq (1 - t_n) \|z_n - x_n\| + Ng(M) r_n \xi(\gamma_{1,2} \sqrt{h_E(\delta_n)}), \quad (3.18)$$

where $M = \max\{M_1, M_2, \dots, M_N\} < +\infty$ and $\gamma_{1,2} = \max_{i=1,2,\dots,N} \{K_{1,i} \sqrt{K_{2,i} L}\}$.

By the above assumption and Lemma 3.2, we conclude that $\|z_n - x_n\| \rightarrow 0$. In addition, by Theorem 3.6,

$$\|z_n - Q_S u\| \leq \|z_n - x_n\| + \|x_n - Q_S u\| \rightarrow 0, \text{ as } n \rightarrow \infty, \quad (3.19)$$

which implies that $\{z_n\}$ converges strongly to $Q_S u$. \square

Finally, in this article we give a method to solve the following problem:

$$\text{Finding an element } x^* \in S = \bigcap_{i=1}^N \text{Fix}(T_i), \quad (3.20)$$

where $T_i : C_i \rightarrow E$, $i = 1, 2, \dots, N$ is nonexpansive nonself-mapping and C_i is a closed convex sunny nonexpansive retract of E .

Lemma 3.8. [14] *Let C be a closed convex subset of a strictly convex Banach space E and let T be a nonexpansive mapping from C into E . Suppose that C is a sunny nonexpansive retract of E . If $\text{Fix}(T) \neq \emptyset$, then $\text{Fix}(T) = \text{Fix}(Q_C T)$, where Q_C is a sunny nonexpansive retraction from E onto C .*

We have the following result:

Theorem 3.9. *Suppose that E is a uniformly convex and uniformly smooth Banach space which has a weakly sequentially continuous normalized duality mapping j from E into E^* . Let C_i be a convex closed sunny nonexpansive retract of E and let $T_i : C_i \rightarrow E$, $i = 1, 2, \dots, N$ be nonexpansive mappings such that $S = \bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. If the sequences $\{r_n\} \subset (0, +\infty)$ and $\{t_n\} \subset (0, 1)$ satisfy*

- (i) $\lim_{n \rightarrow \infty} t_n = 0$; $\sum_{n=0}^{\infty} t_n = +\infty$;
- (ii) $\lim_{n \rightarrow \infty} r_n = +\infty$,

then the sequence $\{u_n\}$ defined by

$$r_n \sum_{i=1}^N f_i(u_{n+1}) + u_{n+1} = t_n u + (1 - t_n) u_n, \quad u, u_0 \in E, \quad n \geq 0, \quad (3.21)$$

converges strongly to $Q_S u$, where Q_S is a sunny nonexpansive retraction from E onto S and $f_i = I - Q_{C_i} T_i Q_{C_i}$, $i = 1, 2, \dots, N$.

Proof. By the Lemma 3.5 and Lemma 3.8, $S = \bigcap_{i=1}^N \text{Fix}(T_i) = \bigcap_{i=1}^N \text{Fix}(f_i)$. Applying Theorem 3.4, we obtain the proof of this Theorem. \square

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Authors' contributions

JKK conceived the study and participated in its design and coordination. JKK suggested many good ideas that are useful for achievement this paper and made the revision. TMT and JKK prepared the manuscript initially and performed all the steps of proof in this research. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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